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AN APPLICATION OF DIVERGING INTEGRALS IN PROBLEMS OF POTENTIAL THEORY AND THE THEORY OF ELASTICITY†

P. I. PERLIN

Moscow

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The use of the method of potentials to construct eigenfunctions in cones for problems of potential theory and the theory of elasticity led to one-dimensional integral equations of the second kind with kernels in the form of integrals [1]. Hence, the conditions for their convergence necessitated introducing limitations on the permissible values of the exponents in the asymptotic form of the solutions (they must be less than unity). In this paper we propose to eliminate these limitations by using regularization (in the sense employed in the theory of generalized functions [2]) of these diverging integrals. Examples of calculations of model problems are given. The algorithm for calculating the kernel of the integral equations obtained is also modified (in general), taking into account the nature of the asymptotic form of the integrand at infinity.

WE WILL consider the Neumann problem as an example. Suppose we have a cone bounded by the surface $\theta=\theta(\varphi)$, $0\leqslant\varphi<2\pi$ and $r>0$. It is required to determine in it a harmonic function, whose normal derivative is zero. We will start form the representation of the eigenfunction in the form $r^\lambda u(\varphi,\ \theta)$ (the case when there are associated functions can be investigated similarly). Then, we obtain the representation in the form $r^\lambda U(\varphi)$ on the surface of the cone for contraction of the eigenfunction.

The function $U(\varphi)$ satisfies a Fredholm integral equation of the second kind

$$r^\lambda U(\varphi)+\frac{1}{2\pi}\int_S r_1^\lambda U(\varphi_1)\frac{d}{dn_{q_1}}\frac{1}{R(q,q_1)}dS_{q_1}=0,\quad R=|q-q_1| \tag{1}$$

Here S is the surface of the cone, q and q_1 are points with coordinates φ, $r(\varphi)$ and $\theta(\varphi)$, and φ_1, r_1 and $\theta(\varphi_1)$. The relationship $r(\varphi)$ also defines the contour on which we require the equation to be satisfied. This contour can be chose fairly arbitrarily, but it is best that the discrete set of points used for the numerical realization for a specified contour should not have common points with the points introduced when evaluating the integrals. We will rewrite Eq. (1) in the following expanded form

$$r^{\lambda-1}U(\varphi)+\frac{1}{2\pi}\int_0^{2\pi}U(\varphi_1)I_1(\varphi,\varphi_1)\int_0^\infty r_1^{\lambda-2}I_2(\varphi_1,r_1)\,dr_1\,d\varphi_1=0 \tag{2}$$

$$I_1(\varphi,\varphi_1)=\cos\vartheta(\varphi_1)\,[\sin\vartheta(\varphi)\cos\vartheta(\varphi)\cos(\varphi_1-\varphi)-\cos\vartheta(\varphi)\sin\vartheta(\varphi_1)]$$

$$I_2(\varphi_1,r_1)=[1-2I_3(\varphi,\varphi_1)\frac{r}{r_1}+\frac{r^2}{r_1^2}]^{-3/2}$$

$$I_3(\varphi,\varphi_1)=\sin\vartheta(\varphi)\sin\vartheta(\varphi_1)\cos(\varphi_1-\varphi)+\cos\vartheta(\varphi)\cos\vartheta(\varphi_1)$$

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Since the asymptotic form of the integrand of the inner integral at infinity has the form $r^{\lambda-2}$, it is natural to include for the existence of regularization considerations [2, p. 97] relating to the integral of that function. On the semiaxis from zero to infinity we will arbitrarily introduce an intermediate point a and the integral is decomposed into two integrals. Then

$$\int_0^\infty r^\mu dr = \int_0^a r^\mu dr + \int_a^\infty r^\mu dr \tag{3}$$

When $\mathrm{Re}\,\mu > -1$ the first integral exists and is equal to $a^{\mu+1}/(\mu+1)$. The function obtained is analytic over the whole plane of the complex variable, with the exception of the point $\mu = -1$.

The second integral exists provided $\mathrm{Re}\,\mu < -1$, it is equal to $-a^{\mu+1}/(\mu+1)$, and is also a function that is analytic over the whole plane, with the exception of the point $\mu = -1$. Summing, we arrive at the fact that the integral (3) is equal to zero for any value of μ, which enables us to convert the inner integral in Eq. (2) to the form

$$\int_0^\infty r_1^{\lambda-2} \xi(\varphi_1, r_1)\, dr_1, \quad \xi(\varphi_1, r_1) = I_2(\varphi_1, r_1) - 1 \tag{4}$$

It is obvious that when $1 < \lambda < 2$ the integral is convergent. Hence, the regularization procedure in fact reduces to subtracting from the integrand the term which gives rise to divergence. In a similar way we can ensure that a convergent representation is obtained for large values of λ. To do this it is necessary to obtain an expansion at infinity of the function $I_2(\varphi_1, r_1)$ for the right number of terms.

The convergent representation in the range $2 < \lambda < 3$ will have the form

$$\int_0^\infty r_1^{\lambda-2} \xi_1(\varphi_1, r_1)\, dr_1, \quad \xi_1(\varphi_1, r_1) = \xi(\varphi_1, r_1) - 3I_3(\varphi, \varphi_1)\frac{r}{r_1} \tag{5}$$

We will present the results of some calculations of test problems whose solutions were obtained [3] by the method of separation of variables. We considered the problem for a circular cone, and to shorten the calculations we used the dependence of the solution on the angular coordinate established there, and it was therefore sufficient to require that the integral equation need be satisfied only at one point of the contour ($r = 1$, $\varphi = 0$). The solution of the problem was then reduced to determining the value of λ for which the integral in (2) was equal to -2π.

We considered two solutions of the Neumann problems $r^{1.245} P_{1.245}(\cos\vartheta)$ and $r^{2.905} \cos 3\varphi P^3_{2.905}(\cos\varphi)$ we obtain the values of the exponents 1.22 and 2.92 by calculation.

A solution of Dirichlet problems was obtained in a similar way. We used equations obtained on the basis of the potential of a double layer. In this case, the integral equation differed from (1) only in the sign in front of the integral. We considered two solutions of the Dirichlet problems: $r^{1.245} \cos\varphi P^1_{1.245}(\cos\theta)$ and $r^{2.136} \cos 2\varphi P^2_{2.136}(\cos\vartheta)$. The values of the exponents 1.22 and 2.10 were obtained by calculation.

The solutions of the Neumann problem with an exponent 2.92, and equally with exponent 0.86, involved certain computational difficulties. To achieve the proper accuracy the integration had to be carried out over the generatrix on very large sections, since the integrand decreased slowly. A modification of the algorithm is proposed (irrespective of the fact of regularization, since we are dealing with converging integrals), which consists of taking into account the asymptotic form of the integrand at infinity. Suppose it is required to satisfy the integral equation at a certain point. When calculating the inner integral (from zero to infinity) we choose a certain point on the generatrix whose distance from the vertex of the cone is of the same order as the point at which the integral is evaluated, and we integrate from zero to this point and from it to infinity. We leave the first integral unchanged, while in the second we add and subtract the following coefficient of its expansion in negative powers of the radius to the factor with degree r_1. The term with the plus sign forms a separate integral, which can be evaluated explicitly. Obviously the remaining integral can be calculated more effectively.

We will give the final expression for the integral in the range $2 < \lambda < 3$

$$\int_0^\infty r_1^{\lambda-2}\xi_1(\varphi_1, r_1)\,dr_1 = \int_0^a r_1^{\lambda-2}\xi_1(\varphi_1, r_1)\,dr_1 + \int_a^\infty r_1^{\lambda-2}\xi_2(\varphi_1, r_1)\,dr_1 - I_4(\varphi, \varphi_1)\frac{r^2 a^{\lambda-3}}{\lambda-3}$$

$$\xi_2(\varphi_1, r_1) = \xi_1(\varphi_1, r_1) - I_4(\varphi, \varphi_1)\frac{r^2}{r_1^2}, \quad I_4(\varphi, \varphi_1) = \frac{15 I_3^2(\varphi, \varphi_1) - 3}{2} \tag{6}$$

We carried out calculations for values of the exponents of 0.5 and 0.9. The integrals were evaluated at the point at unit distance from the vertex of the cone, and the auxiliary point introduced above was placed at a distance of 10. By considering the asymptotic form an accuracy was achieved to the third place in the first case in a section of length 10, and in the second in a section of length 20. At the same time, calculations carried out ignoring the asymptotic form in the first case required integration over a section of length 3×10^4. In the second case, integration over a section of length 10^6 led to an error of 35%.

Hence, the results of the calculations confirm the effectiveness of the modification, particularly for values of the exponents close to integer numbers but less than them.

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